$\mathbb{Z}_9$ Flavour Dynamics
A Lagrangian Realization

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Soli Deo gloria.

Abstract

We construct an explicit Froggatt–Nielsen flavour model based on a discrete $\mathbb{Z}_9$ symmetry that realizes the mass and mixing predictions of the companion paper [1]. A single flavon field ϕ with unit $\mathbb{Z}_9$ charge and expansion parameter $\varepsilon = \langle\phi\rangle/\Lambda = 2/9$ — the ratio of the $\mathbb{Z}_9$ generator to the modulus, derived from group structure rather than fitted — reproduces all nine charged fermion masses with $O(1)$ Yukawa coefficients in the range $[0.49, 2.09]$, generates the CKM mixing hierarchy, accommodates large PMNS mixing angles through a type-I seesaw mechanism, and solves the Strong CP problem by ensuring $\arg(\det M_q) = 0$ identically. The $\mathbb{Z}_9$-invariant scalar potential $V(\phi)$ admits nine degenerate vacua and, when the symmetry is gauged via the Krauss–Wilczek mechanism, confines the associated domain walls. All discrete anomalies are cancelled by a Green–Schwarz mechanism. The model extends the Standard Model by one scalar field and one discrete symmetry, with no continuous flavour symmetry, no additional fermions beyond right-handed neutrinos, and no supersymmetry.

1 Introduction

The Standard Model of particle physics is a spectacularly successful quantum field theory with one embarrassing feature: 19 of its parameters are unexplained. Masses, coupling constants, and mixing angles are measured to extraordinary precision but are not derived from any deeper principle. They are inputs, not outputs. The companion paper [1] showed that a single algebraic structure — the integers modulo 9 under multiplication — encodes 32 fundamental quantities spanning 12 orders of magnitude. The fine structure constant ($1/\alpha = 137$), the weak mixing angle ($\sin^2\theta_W = 2/9$), the strong coupling ($\alpha_s = 2/17$), all nine charged fermion masses, the electroweak vacuum expectation value, the CKM and PMNS mixing matrices, and multiple cosmological parameters all follow from four $\mathbb{Z}_9$ structural constants: the generator $g = 2$, the depth $N = 8$, the axis (modulus) $n = 9$, and the depth factor $2N + 1 = 17$. The statistical significance was estimated conservatively at 10^{-65}. Those results are arithmetic. They derive numbers from the multiplicative structure of a finite ring. The present paper answers the dynamical question: can these values emerge from a quantum field theory Lagrangian with $\mathbb{Z}_9$ as its flavour symmetry? We show that they can. The construction is a Froggatt–Nielsen (FN) model [4] with a discrete $\mathbb{Z}_9$ symmetry. Froggatt–Nielsen models explain fermion mass hierarchies by assigning charges under a flavour symmetry and generating Yukawa couplings as powers of a small expansion parameter $\varepsilon = \langle\phi\rangle/\Lambda$, where ϕ is a flavon field. In conventional FN models, the symmetry is a continuous $U(1)$ and ε is fitted to the Cabibbo angle. In our model, the symmetry is exactly $\mathbb{Z}_9$ and $\varepsilon = 2/9$ is algebraically forced: it is the ratio of the $\mathbb{Z}_9$ generator (2) to the modulus (9). This is not a choice. It is the unique structure constant that measures one step of the generator relative to the full group.

The paper is organized as follows. Section 2 reviews the $\mathbb{Z}_9$ algebraic structure. Section 3 presents the field content and charge assignments. Section 4 constructs the Yukawa sector and derives the mass matrices. Section 5 extracts the fermion mass eigenvalues. Section 6 derives the CKM matrix. Section 7 extends to the lepton sector and PMNS matrix via a type-I seesaw. Section 8 addresses the Strong CP problem. Section 9 constructs the scalar potential. Section 10 performs the anomaly analysis. Section 11 discusses gauging $\mathbb{Z}_9$ and domain wall confinement. Section 12 presents the coupling constants. Section 13 summarizes and identifies open questions for [2].

2 $\mathbb{Z}_9$ Algebraic Structure

We recall the essential structure established in [1]. The ring $\mathbb{Z}_9 = \{0, 1, 2, \ldots, 8\}$ under multiplication modulo 9 decomposes into two complementary structures:

$$\text{Multiplicative group: } \mathbb{Z}_9^* = \{1, 2, 4, 5, 7, 8\}, \quad |\mathbb{Z}_9^*| = \phi(9) = 6 \tag{1}$$

$$\text{Multiplicative ideal (axis): } I = \{0, 3, 6\} \tag{2}$$

The group $\mathbb{Z}_9^*$ is cyclic of order 6, generated by $g = 2$, which is the smallest element generating the full group. The generator orbit is:

$$2^1 = 2, \quad 2^2 = 4, \quad 2^3 = 8, \quad 2^4 = 7, \quad 2^5 = 5, \quad 2^6 = 1 \pmod 9 \tag{3}$$

The four structural constants that determine all physics are:

Constant	Symbol	Value	Origin
Generator	g	2	Smallest generator of $\mathbb{Z}_9^*$
Depth	N	8	$\lvert\mathbb{Z}_9^*\rvert + \lvert I\rvert - 1 = 6 + 3 - 1$
Modulus	n	9	Unique root of $2n^2 - 3n + 2 = 137$
Depth factor	$2N + 1$	17	Controls strong sector

From these:

$$1/\alpha = N(2N + 1) + 1 = 8 \times 17 + 1 = 137 \tag{4}$$

$$\sin^2\theta_W = g/n = 2/9 \approx 0.2222 \tag{5}$$

$$\alpha_s = g/(2N + 1) = 2/17 \approx 0.1176 \quad \text{at } M_Z \tag{6}$$

2.1 Uniqueness

The modulus $n = 9$ is unique. The discriminant of $2n^2 - 3n + 2 = 137$ is $\Delta = 9 - 8(2 - 137) = 1089 = 33^2$. With discriminant a perfect square, the equation has rational roots $n = (3 \pm 33)/4$, giving $n = 9$ (the positive integer root) and $n = -15/2$ (discarded). No other modulus produces the correct fine structure constant.

2.2 The Expansion Parameter

The ratio $\varepsilon = g/n = 2/9$ is the $\mathbb{Z}_9$ structure constant that measures one generator step relative to the full group. In a Froggatt–Nielsen model, this becomes the expansion parameter of the Yukawa sector. The Cabibbo angle prediction $\sin\theta_C = \varepsilon = 2/9 = 0.2222$ matches the measured value 0.2243 to 0.9%. This is a prediction, not a fit: any other value of ε would break the Cabibbo angle, and any other modulus would break α.

3 Field Content and Charge Assignments

3.1 Particle Content

The model is the Standard Model extended by one complex scalar ϕ (the flavon) and three right-handed neutrinos ν_{R_i}. The field content under $SU(3)_C \times SU(2)_L \times U(1)_Y$ is standard. The only new quantum number is a $\mathbb{Z}_9$ charge carried by every field.

$$\text{Higgs doublet } H: \quad \mathbb{Z}_9 \text{ charge} = 9 \tag{7}$$
$$\text{Flavon } \phi: \quad \mathbb{Z}_9 \text{ charge} = +1 \tag{8}$$

The Higgs is neutral under $\mathbb{Z}_9$ — it breaks electroweak symmetry but not flavour symmetry. The flavon carries unit charge and breaks $\mathbb{Z}_9$ when it acquires a VEV.

3.2 Fermion Charge Assignments

The $\mathbb{Z}_9$ charges for all Standard Model fermions are:

Field	$SU(3) \times SU(2) \times U(1)$	Gen 1	Gen 2	Gen 3
Q_L (quark doublet)	$(3, 2, +1/6)$	3	2	0
u_R (up singlet)	$(3, 1, +2/3)$	5	1	0
d_R (down singlet)	$(3, 1, -1/3)$	4	3	2
L_L (lepton doublet)	$(1, 2, -1/2)$	3	2	0
e_R (charged lepton)	$(1, 1, -1)$	5	3	3
ν_R (right-handed ν)	$(1, 1, 0)$	6	7	0
H (Higgs)	$(1, 2, +1/2)$	9	—	—
ϕ (flavon)	$(1, 1, 0)$	1	—	—

3.3 Principles Constraining the Charges

The charges are not arbitrary. They satisfy three structural requirements:

1. $SU(5)$ **compatibility.** The left-handed charges satisfy $q(Q_{L_i}) = q(L_{L_i})$ for all generations. This allows Q_L and L_L to sit in the same $\bar{5}$ representation, consistent with $SU(5)$ multiplet structure (though $\mathbb{Z}_9$ itself is fundamental, not derived from GUT breaking—see [2]).

2. **Third-generation minimality.** The third generation carries charge 0 for Q_L, u_R, and ν_R. This ensures the top quark Yukawa is unsuppressed ($\varepsilon^0 = 1$), so $m_t \sim v = 174$ GeV naturally.

3. **Hierarchy depth** $= N$**.** The maximum charge separation across all sectors is $8 = N$. The full mass hierarchy $m_t/m_e = 3.4 \times 10^5$ requires $(9/2)^{8.5} \approx 3.4 \times 10^5$. A ring with fewer than 9 elements could not accommodate the observed hierarchy; a larger ring would leave unused charge space.

4 The Yukawa Lagrangian

4.1 General Structure

The most general $\mathbb{Z}_9$-invariant Yukawa Lagrangian couples fermions to the Higgs through powers of the flavon:

$$\begin{aligned}\mathcal{L}_Y = &\sum_{ij} c^u_{ij} \left(\frac{\phi}{\Lambda}\right)^{n^u_{ij}} \bar{Q}_{L_i} \tilde{H} u_{R_j} + \sum_{ij} c^d_{ij} \left(\frac{\phi}{\Lambda}\right)^{n^d_{ij}} \bar{Q}_{L_i} H d_{R_j} \\ &+ \sum_{ij} c^e_{ij} \left(\frac{\phi}{\Lambda}\right)^{n^e_{ij}} \bar{L}_{L_i} H e_{R_j} + \sum_{ij} c^\nu_{ij} \left(\frac{\phi}{\Lambda}\right)^{n^\nu_{ij}} \bar{L}_{L_i} \tilde{H} \nu_{R_j} + \text{h.c.}\end{aligned} \tag{9}$$

Each power is fixed by $\mathbb{Z}_9$ charge conservation:

$$n^f_{ij} = (q_{L_i} + q_{R_j}) \bmod 9 \tag{10}$$

The c^f_{ij} are dimensionless $O(1)$ complex coefficients — the only free parameters in the Yukawa sector. After the flavon acquires its VEV, $\langle\phi\rangle/\Lambda = \varepsilon = 2/9$, each entry is suppressed by ε^n. The mass matrices are

$$M^f_{ij} = c^f_{ij} \varepsilon^{n_{ij}} v \tag{11}$$

4.2 Power Matrices

Computing $n_{ij} = (q_{L_i} + q_{R_j}) \bmod 9$ for each sector:

Up-type quarks (Q_L charges: 3, 2, 0; u_R charges: 5, 1, 0):

$$\begin{pmatrix} u_R(5) & c_R(1) & t_R(0) & \\ Q_1(3) & 8 & 4 & 3 \\ Q_2(2) & 7 & 3 & 2 \\ Q_3(0) & 5 & 1 & 0 \end{pmatrix} \tag{12}$$

Down-type quarks (Q_L charges: 3, 2, 0; d_R charges: 4, 3, 2):

$$\begin{pmatrix} d_R(4) & s_R(3) & b_R(2) & \\ Q_1(3) & 7 & 6 & 5 \\ Q_2(2) & 6 & 5 & 4 \\ Q_3(0) & 4 & 3 & 2 \end{pmatrix} \tag{13}$$

Charged leptons (L_L charges: 3, 2, 0; e_R charges: 5, 3, 3):

$$\begin{pmatrix} e_R(5) & \mu_R(3) & \tau_R(3) & \\ L_1(3) & 8 & 6 & 6 \\ L_2(2) & 7 & 5 & 5 \\ L_3(0) & 5 & 3 & 3 \end{pmatrix} \tag{14}$$

4.3 Texture Structure

Several features of the power matrices are noteworthy:

- The up-type $(3,3)$ entry is $\varepsilon^0 = 1$: the top quark Yukawa is unsuppressed.
- The down-type $(3,3)$ entry is ε^2: the bottom quark is suppressed by two powers, naturally explaining $m_b/m_t \sim \varepsilon^2 \approx 1/20$.
- The charged lepton matrix has a $(2,3) = (3,3)$ degeneracy (both equal ε^3), reflecting $q(\mu_R) = q(\tau_R) = 3$. This produces $m_\tau/m_\mu \sim \varepsilon^2$ rather than ε^3, consistent with the relatively mild τ/μ mass ratio.

5 Fermion Mass Eigenvalues

5.1 Leading-Order Masses

At leading order (diagonal dominance), the fermion masses are $m_f \approx c_{ii}\varepsilon^{n_{ii}}v$. With $v = 174$ GeV and $\varepsilon = 2/9$:

5.2 Assessment of $O(1)$ Coefficients

All nine $O(1)$ coefficients lie in the range $[0.49, 2.09]$. The mean is 1.02, the geometric mean is 0.89, and the standard deviation is 0.46. No coefficient deviates from unity by more than a factor of 2.1. This is standard Froggatt–Nielsen quality: a successful texture model is defined as one where all $O(1)$ coefficients are genuinely of order 1, meaning no coefficient exceeds a factor of ~ 3. The present model satisfies this criterion comfortably. For comparison,

Fermion	Diagonal power	ε^n	$\varepsilon^n \times v$ (GeV)	Measured (GeV)	$O(1)$ coeff c_{ii}
t (top)	0	1.000	174.0	172.69 ± 0.30	0.993
c (charm)	3	1.097×10^{-2}	1.909	1.270 ± 0.020	0.665
u (up)	8	5.947×10^{-6}	1.035×10^{-3}	2.16×10^{-3}	2.09
b (bottom)	2	4.938×10^{-2}	8.593	4.180 ± 0.030	0.486
s (strange)	5	5.419×10^{-4}	9.429×10^{-2}	9.34×10^{-2}	0.991
d (down)	7	2.676×10^{-5}	4.657×10^{-3}	4.67×10^{-3}	1.003
τ (tau)	3	1.097×10^{-2}	1.909	1.777	0.931
μ (muon)	5	5.419×10^{-4}	9.429×10^{-2}	0.1057	1.121
e (electron)	8	5.947×10^{-6}	1.035×10^{-3}	5.110×10^{-4}	0.494

the best-known FN models in the literature [11, 12] typically achieve $O(1)$ coefficients in the range $[0.3, 3]$. The $\mathbb{Z}_9$ model's range $[0.49, 2.09]$ is tighter than most published models, suggesting that the $\mathbb{Z}_9$ charge assignments are better than average at capturing the true mass pattern.

5.3 Hierarchy Architecture

The diagonal powers encode the hierarchy architecture of each sector:

$$\text{Up-type: } (8,3,0) \quad \Rightarrow \quad \text{gaps} = (5,3) \tag{15}$$

$$\text{Down-type: } (7,5,2) \quad \Rightarrow \quad \text{gaps} = (2,3) \tag{16}$$

$$\text{Leptons: } (8,5,3) \quad \Rightarrow \quad \text{gaps} = (3,2) \tag{17}$$

The total depth from the lightest fermion (electron, power 8) to the heaviest (top, power 0) is exactly $N = 8$, the number of non-trivial multiplicative elements in $\mathbb{Z}_9$ ($|\mathbb{Z}_9^*| + |I| - 1 = 6 + 3 - 1 = 8$). This is the hierarchy depth theorem of [1]: the ring controls how many decades of mass range are available.

6 The CKM Matrix

6.1 Mixing from Misalignment

The Cabibbo–Kobayashi–Maskawa matrix $V_{CKM} = U_L^{u\dagger} U_L^d$ arises from the misalignment between the unitary matrices that diagonalize the up-type and down-type mass matrices. In the FN framework, the diagonalization matrices are approximately controlled by the charge differences between generations:

$$|V_{ij}| \sim \varepsilon^{|q(Q_{L_i}) - q(Q_{L_j})|} \quad \text{for } i \neq j \tag{18}$$

With $q(Q_L) = (3, 2, 0)$:

$$|V_{us}| \sim \varepsilon^{|3-2|} = \varepsilon^1 = 2/9 = 0.2222 \tag{19}$$

$$|V_{cb}| \sim \varepsilon^{|2-0|} = \varepsilon^2 = 4/81 = 0.0494 \tag{20}$$

$$|V_{ub}| \sim \varepsilon^{|3-0|} = \varepsilon^3 = 8/729 = 0.01097 \tag{21}$$

6.2 Comparison with Measurement

Element	Z$_9$ FN prediction	Refined Z$_9$ [1]	Measured (PDG)	FN error
$\|V_{us}\|$	$\varepsilon = 0.2222$	$2/9 = 0.2222$	0.2243 ± 0.0005	-0.9%
$\|V_{cb}\|$	$\varepsilon^2 = 0.0494$	$2/49 = 0.0408$	0.0405 ± 0.0012	$+22\%$
$\|V_{ub}\|$	$\varepsilon^3 = 0.01097$	$2/441 = 0.0045$	0.00382 ± 0.00020	$+188\%$

The FN predictions (column 2) are accurate to the expected FN precision: one $O(1)$ factor per power of ε. The refined $\mathbb{Z}_9$ arithmetic predictions from [1] (column 3) are substantially more precise, achieving sub-percent accuracy for $|V_{us}|$. This indicates that the full $\mathbb{Z}_9$ arithmetic captures structure beyond leading-order FN, presumably through the higher-order corrections encoded in the off-diagonal entries.

6.3 The Cabibbo Coincidence Resolved

The near-equality $\sin\theta_C \approx \sin^2\theta_W$ has been noted since the 1970s [5] but never explained. In this framework, both equal 2/9: the Cabibbo angle is $\varepsilon^1 = 2/9$, and the weak mixing angle is $g/n = 2/9$. They are the same $\mathbb{Z}_9$ structure constant appearing in different physical roles — one in the Yukawa sector (mass mixing), one in the gauge sector (coupling unification). The "coincidence" is structural identity.

7 Lepton Sector and PMNS Matrix

7.1 The Problem of Large Mixing

The PMNS neutrino mixing matrix exhibits two near-maximal mixing angles (θ_{12}, θ_{23}) and one moderate angle ($\theta_{13} \approx 8.5$). This is qualitatively different from the CKM hierarchy of small angles. In Froggatt–Nielsen models, large neutrino mixing requires special structure in the neutrino mass matrix that partially cancels the hierarchical contribution from the charged-lepton sector.

7.2 Dirac Neutrino Mass Matrix

The right-handed neutrinos carry $\mathbb{Z}_9$ charges $\nu_R = (6, 7, 0)$. The Dirac neutrino Yukawa powers are $n^{\nu}_{ij} = (q(L_{L_i}) + q(\nu_{R_j})) \bmod 9$:

$$
\begin{pmatrix}
\nu_{R_1}(6) & \nu_{R_2}(7) & \nu_{R_3}(0) & \\
L_1(3) & 0 & 1 & 3 \\
L_2(2) & 8 & 0 & 2 \\
L_3(0) & 6 & 7 & 0
\end{pmatrix} \tag{22}
$$

The critical feature is the pair of zeros: M_{11}^D and M_{22}^D are unsuppressed ($\varepsilon^0 = 1$). This occurs because $3 + 6 = 9 \equiv 0$ and $2 + 7 = 9 \equiv 0 \pmod 9$. The $\mathbb{Z}_9$ charges of L_L and ν_R are chosen so that the first two generations have complementary charges summing to the modulus. This is the algebraic origin of large neutrino mixing: the first two generations are "paired" in the Dirac sector.

7.3 Majorana Mass Matrix

The right-handed neutrinos have a Majorana mass matrix with powers $n_{ij}^M = (q(\nu_{R_i}) + q(\nu_{R_j})) \bmod 9$:

$$
\begin{pmatrix}
\nu_{R_1}(6) & \nu_{R_2}(7) & \nu_{R_3}(0) & \\
\nu_{R_1}(6) & 3 & 4 & 6 \\
\nu_{R_2}(7) & 4 & 5 & 7 \\
\nu_{R_3}(0) & 6 & 7 & 0
\end{pmatrix} \tag{23}
$$

The $(3,3)$ entry is unsuppressed, while the $(1,1)$ entry goes as ε^3 and the $(1,2)$ entry as ε^4. The Majorana mass scale is a free parameter $M_R \sim \Lambda$, with the seesaw formula $m_\nu = -M_D M_R^{-1} M_D^T$ generating light neutrino masses.

7.4 Seesaw and Large Mixing

The seesaw mechanism combines the hierarchical M_R with the partially democratic M_D. The key: because M_{11}^D and M_{22}^D are both $O(v)$ (unsuppressed), the light neutrino mass matrix inherits a near-democratic structure in the $(1,2)$ block. After charged-lepton diagonalization contributes additional small rotations, the resulting PMNS matrix has:

- Two large angles (θ_{12}, θ_{23}): from the democratic $(1,2)$ structure of M_D.
- One moderate angle ($\theta_{13} \sim \varepsilon$): suppressed by one power of ε relative to the large angles, because the $(1,3)$ Dirac entry is ε^3 while the $(1,1)$ and $(2,2)$ entries are ε^0.
- Normal ordering with $m_1 \approx 0$: the seesaw rank structure generically produces one very light eigenvalue when the Dirac matrix has a specific texture-zero pattern.

7.5 PMNS Predictions

The $\mathbb{Z}_9$ arithmetic predictions from [1] for the PMNS parameters are:
The FN Lagrangian provides the dynamical mechanism: the mixing angles arise from the interplay between the hierarchical charged-lepton mass matrix and the seesaw-generated neutrino mass matrix. The mass-squared ratio $\Delta m_{31}^2/\Delta m_{21}^2 = 34 = 2 \times 17$ (generator $\times$

Parameter	$\mathbb{Z}_9$ prediction [1]	Measured [12]	Error
$\sin^2\theta_{12}$	$4/13 = 0.3077$	0.307 ± 0.013	$+0.2\%$
$\sin^2\theta_{13}$	$1/45 = 0.02222$	0.0220 ± 0.0007	$+1.0\%$
$\sin^2\theta_{23}$	$4/7 = 0.5714$	0.546–0.572	within 1σ
δ_{CP}	$10\pi/9 = 200$	197 ± 25	$+1.5\%$
$\Delta m^2_{31}/\Delta m^2_{21}$	$34 = 2 \times 17$	33.6 ± 0.9	$+1.3\%$
m_1	0	unknown	testable

depth factor) is one of the most striking $\mathbb{Z}_9$ predictions, and the zero lightest mass $m_1 = 0$ is testable by next-generation experiments (KATRIN, Project 8, neutrinoless double-beta decay).

8 The Strong CP Problem

8.1 Statement of the Problem

The QCD Lagrangian contains the CP-violating parameter $\bar{\theta} = \theta_{QCD} + \arg(\det M_q)$, where θ_{QCD} is the bare vacuum angle and M_q is the quark mass matrix. Neutron electric dipole moment measurements constrain $|\bar{\theta}| < 10^{-10}$. The Standard Model provides no explanation for this extreme smallness — this is the Strong CP problem.

8.2 Part 1: Algebraic Solution ($\arg\det M_q = 0$)

The $\mathbb{Z}_9$ framework solves the second term — $\arg(\det M_q) = 0$ — by construction. The quark mass matrices take the form $M^f_{ij} = c^f_{ij}\varepsilon^{n_{ij}}v$. The determinant is:

$$\det M_q = \det M_u \times \det M_d = v^6 \times \varepsilon^{\sum n^u} \times \varepsilon^{\sum n^d} \times (\text{product of } O(1) \text{ coefficients}) \qquad (24)$$

The total FN suppression is $\varepsilon^{11+14} = \varepsilon^{25}$, which is real and positive ($\varepsilon = 2/9 > 0$). In the $\mathbb{Z}_9$ arithmetic of [1], every fermion mass is a rational number — a ratio of integers composed from the vocabulary $\{2, 5, 8, 9, 17\}$. Rational numbers have zero imaginary part. Therefore $\arg(\det M_q) = 0$ identically. This is not an assumption — it is a structural consequence. In a general FN model, the $O(1)$ coefficients can be complex, and $\arg(\det M_q)$ is generically $O(1)$. The $\mathbb{Z}_9$ framework constrains these coefficients to be real because the underlying arithmetic is over the integers.

8.3 Part 2: Topological Argument ($\theta_{QCD} = 0$)

The bare vacuum angle θ_{QCD} parameterizes the circle $S^1 = [0, 2\pi)$. If the underlying angular structure is $\mathbb{Z}_9$ rather than continuous $U(1)$, then θ is restricted to the nine values $\theta \in \{2\pi k/9 : k = 0, 1, \ldots, 8\}$. CP invariance selects $\theta = 0$ or $\theta = \pi$. Among the nine allowed values, only $\theta = 0$ is CP-conserving. The $\mathbb{Z}_9$ perspective gives this a group-theoretic interpretation: the complete generator cycle $2^6 \equiv 1 \pmod 9$ returns to the identity. The full group cycle (9 elements traversed) is $9 \equiv 0 \pmod 9$. Completeness is emptiness. The vacuum does not choose $\theta = 0$ by fine-tuning — it arrives at 0 because a complete $\mathbb{Z}_9$ cycle has nowhere else to go.

We emphasize that Part 1 ($\arg\det M_q = 0$) is a theorem within the model. Part 2 ($\theta_{QCD} = 0$) is a motivated conjecture that requires a rigorous discrete gauge theory derivation (Section 11). The combination $\bar{\theta} = 0{+}0 = 0$ solves the Strong CP problem without an axion.

9 The Scalar Potential

9.1 Construction

The flavon ϕ carries $\mathbb{Z}_9$ charge $+1$. The available invariants for the potential are: $|\phi|^2$ (charge 0, renormalizable), $|\phi|^4$ (charge 0, renormalizable), and $\phi^9 + \phi^{*9}$ (the lowest-order $\mathbb{Z}_9$-specific invariant, dimension 9). The most general leading-order potential is:

$$V(\phi) = -\mu^2|\phi|^2 + \lambda|\phi|^4 + \frac{g_9}{\Lambda^5}(\phi^9 + \phi^{*9}) \tag{25}$$

where $\mu^2 > 0$ triggers spontaneous breaking, $\lambda > 0$ ensures stability, and g_9 is a dimensionless coupling. The ϕ^9 term is the hallmark of $\mathbb{Z}_9$: it is the lowest-dimension operator that distinguishes $\mathbb{Z}_9$ from $U(1)$. For $\mathbb{Z}_N$ with any other N, this term would be ϕ^N, but only for $N = 9$ does the expansion parameter $g/N = 2/9$ reproduce the observed Cabibbo angle.

9.2 Vacuum Structure

Writing $\phi = \rho e^{i\theta}$, the potential separates:

$$V(\rho, \theta) = -\mu^2\rho^2 + \lambda\rho^4 + \frac{2g_9\rho^9}{\Lambda^5}\cos(9\theta) \tag{26}$$

The radial minimum is at $\langle\rho\rangle = \mu/\sqrt{2\lambda} + O(g_9)$. Setting $\langle\rho\rangle/\Lambda = 2/9$:

$$\mu^2 = 2\lambda(2\Lambda/9)^2 = \frac{8\lambda\Lambda^2}{81} \tag{27}$$

The angular potential $\cos(9\theta)$ has nine degenerate minima at $\theta_k = 2\pi k/9$ for $k = 0, 1, \ldots, 8$ (when $g_9 > 0$). The nine vacua are:

$$\langle\phi\rangle_k = \frac{2\Lambda}{9}e^{2\pi ik/9}, \quad k = 0, 1, \ldots, 8 \tag{28}$$

Physics is identical in all nine vacua (they are related by the broken $\mathbb{Z}_9$ symmetry). The universe selects one during the phase transition.

9.3 Mass Spectrum

After symmetry breaking, expanding $\phi = (2\Lambda/9 + h)e^{i\pi_\phi/f}$:

$$\text{Radial mode } h: \quad m_h^2 = 4\lambda(2\Lambda/9)^2 = \frac{32\lambda\Lambda^2}{81} \tag{29}$$

$$\text{Angular mode } \pi_\phi: \quad m_\pi^2 = 162 g_9 (2/9)^7 \Lambda^2 \tag{30}$$

The radial mode h is a massive scalar (the "flavon Higgs"). The angular mode π_ϕ (the "axiflavon") acquires mass from the ϕ^9 term. If $\mathbb{Z}_9$ is gauged (Section 11), the angular mode is eaten by the discrete gauge field and does not appear as a physical particle.

9.4 Naturalness

The relationship $\varepsilon = 2/9$ is not dynamically derived from the potential — it is an input ($\mu^2 = 8\lambda\Lambda^2/81$ must be arranged). This is the same status as the electroweak hierarchy problem: the Higgs VEV $v = 246$ GeV is also an input of the Standard Model potential. The $\mathbb{Z}_9$ framework does not solve the hierarchy problem, but it reframes it: instead of explaining why v/M_{Pl} is small, one must explain why $\langle\phi\rangle/\Lambda = 2/9$. The algebraic answer (Section 2.2) is that 2/9 is forced by group structure. Whether a dynamical mechanism exists that selects this ratio automatically remains an open question.

10 Anomaly Analysis

10.1 Discrete Anomaly Conditions

For a $\mathbb{Z}_N$ symmetry to be gauged, its mixed anomalies with the Standard Model gauge groups must vanish modulo N (possibly after Green–Schwarz cancellation). The anomaly coefficients are [7, 9]:

$$A[SU(3)^2 \times \mathbb{Z}_N] = 2\sum q(Q_L) + \sum q(u_R) + \sum q(d_R) \tag{31}$$

$$A[SU(2)^2 \times \mathbb{Z}_N] = 3\sum q(Q_L) + \sum q(L_L) \tag{32}$$

$$A[\text{grav}^2 \times \mathbb{Z}_N] = 6\sum q(Q_L) + 3\sum q(u_R) + 3\sum q(d_R) + 2\sum q(L_L) + \sum q(e_R) + \sum q(\nu_R) \tag{33}$$

10.2 Explicit Computation

Substituting the charge assignments from Section 3.2:

$$\sum q(Q_L) = 3 + 2 + 0 = 5 \tag{34}$$
$$\sum q(u_R) = 5 + 1 + 0 = 6 \tag{35}$$
$$\sum q(d_R) = 4 + 3 + 2 = 9 \equiv 0 \pmod 9 \tag{36}$$
$$\sum q(L_L) = 3 + 2 + 0 = 5 \tag{37}$$
$$\sum q(e_R) = 5 + 3 + 3 = 11 \equiv 2 \pmod 9 \tag{38}$$
$$\sum q(\nu_R) = 6 + 7 + 0 = 13 \equiv 4 \pmod 9 \tag{39}$$

$$A[SU(3)^2] = 2(5) + 6 + 0 = 16 \equiv 7 \pmod 9 \tag{40}$$
$$A[SU(2)^2] = 3(5) + 5 = 20 \equiv 2 \pmod 9 \tag{41}$$
$$A[\text{grav}^2] = 30 + 18 + 0 + 10 + 2 + 4 = 64 \equiv 1 \pmod 9 \tag{42}$$

10.3 Resolution: Green–Schwarz Mechanism

None of the anomaly coefficients vanish mod 9. This is standard for discrete flavour symmetries and does not invalidate the model [7, 9]. Two resolutions exist:

1. **Global symmetry.** If $\mathbb{Z}_9$ is a global symmetry, anomaly cancellation is not required. The symmetry is approximate, broken by quantum gravity at $O(M_{\text{Pl}})$, but the Yukawa textures are protected to the accuracy needed.

2. **Green–Schwarz cancellation.** If $\mathbb{Z}_9$ is gauged (as motivated by domain wall confinement), the anomalies are cancelled by a Green–Schwarz shift of an axion-like field [6, 7]. In string compactifications, this mechanism is automatic for discrete remnants of continuous gauge symmetries. The GS mechanism requires a single additional axion field with prescribed couplings to the gauge field strengths, constrained by:

$$\delta_{\text{GS}} a = \frac{2\pi}{9} \sum_i k_i \alpha_i \tilde{F}^i F_i \tag{43}$$

   where the k_i are fixed by the anomaly coefficients. This is the same mechanism employed in heterotic string compactifications with discrete Wilson lines.

11 Gauging $\mathbb{Z}_9$ and Domain Wall Confinement

11.1 The Krauss–Wilczek Mechanism

Krauss and Wilczek [8] showed that discrete symmetries can arise as exact remnants of broken continuous gauge symmetries. The construction: a continuous $U(1)_F$ flavour symmetry is broken to $\mathbb{Z}_9$ at scale Λ by a scalar Φ carrying $U(1)_F$ charge 9. Below the breaking scale, the resulting $\mathbb{Z}_9$ is an exact gauge symmetry, protected against violation by quantum gravity effects. Banks and Seiberg [9] proved that such discrete gauge symmetries are the only discrete symmetries consistent with quantum gravity.

11.2 Domain Wall Problem and Solution

The nine degenerate vacua of the $\mathbb{Z}_9$-breaking potential (Section 9.2) produce domain walls. If $\mathbb{Z}_9$ is a global symmetry, these walls are stable, and their energy density quickly dominates the universe — the cosmological domain wall problem. If $\mathbb{Z}_9$ is gauged (i.e., a remnant of $U(1)_F$), the walls are confined. The broken $U(1)_F$ gauge field produces cosmic strings, and the domain walls connecting different $\mathbb{Z}_9$ vacua become bounded by strings [10]. The string-wall network collapses, and the domain walls annihilate on a timescale set by the string tension, solving the cosmological problem.

11.3 Phenomenological Consequences

In the gauged scenario, the angular mode of ϕ is eaten by the massive $U(1)_F$ gauge boson. The physical spectrum contains:

1. the radial flavon h, with mass $m_h = O(\Lambda)$;
2. the heavy $U(1)_F$ gauge boson, with mass $m_{A'} = g_F \times 9 \times \langle\phi\rangle = 2g_F\Lambda$; and
3. the Green–Schwarz axion, with mass set by the anomaly cancellation conditions.

Flavour-changing neutral currents mediated by the flavon h constrain $\Lambda > O(\text{few TeV})$. The flavon couples to fermion pair (i, j) with strength proportional to $\varepsilon^{|q_i - q_j|}/\Lambda$, producing hierarchical flavour violation that is most visible in the top-charm sector (charge difference 1, coupling $\sim \varepsilon/\Lambda$) and the bottom-strange sector (charge difference 2, coupling $\sim \varepsilon^2/\Lambda$).

12 Coupling Constants

12.1 The Three $\mathbb{Z}_9$ Couplings

The $\mathbb{Z}_9$ framework determines three fundamental coupling constants from its structural constants:

Coupling	$\mathbb{Z}_9$ formula	Predicted	Measured
$1/\alpha$	$N(2N+1)+1 = 137$	137.000	137.036 (+0.026%)
$\sin^2\theta_W$	$1 - 7/9 = 2/9$	0.2222	0.2232 (on-shell, +0.46%)
$\alpha_s(M_Z)$	$g/(2N+1) = 2/17$	0.1176	0.1180 ± 0.0009 (+0.30%)

The weak mixing angle prediction $\sin^2\theta_W = 2/9$ follows algebraically from the $\mathbb{Z}_9$ boson mass predictions in [1]. The Z boson mass is $M_Z = M_W/\cos\theta_W = M_W \times 3/\sqrt{7}$, so $\cos^2\theta_W = 7/9$ and $\sin^2\theta_W = 1 - 7/9 = 2/9$ exactly. In the on-shell renormalization scheme, $\sin^2\theta_W =$

$1 - M_W^2/M_Z^2 = 0.2232$, giving a tree-level prediction error of +0.46%. The larger $\overline{\text{MS}}$ value 0.23122 at M_Z includes Standard Model radiative corrections, as expected for a tree-level prediction.

12.2 Identification of the FN Scale

A natural identification of the FN scale is $\Lambda = m_e \times N^2 \times n \times 5^3 = 36.79$ GeV, since this decomposes entirely into $\mathbb{Z}_9$ vocabulary. The flavon VEV would then be $\langle\phi\rangle = 2\Lambda/9 \sim 8$ GeV, in the B-physics / Υ energy range. This is a plausible but conjectural identification: verifying it requires showing that threshold corrections at $\mu = \Lambda$ are consistent with the measured couplings. This computation remains a target for future work.

13 Summary and Outlook

13.1 What This Paper Establishes

We have constructed an explicit quantum field theory that realizes the $\mathbb{Z}_9$ arithmetic of [1]:

1. A complete Froggatt–Nielsen model with discrete $\mathbb{Z}_9$ flavour symmetry and expansion parameter $\varepsilon = 2/9$ derived from group structure.
2. All nine charged fermion masses reproduced with $O(1)$ coefficients in $[0.49, 2.09]$.
3. CKM mixing hierarchy from $\mathbb{Z}_9$ charge differences: $|V_{us}| = \varepsilon$, $|V_{cb}| = \varepsilon^2$, $|V_{ub}| = \varepsilon^3$.
4. Large PMNS mixing from a type-I seesaw with $\mathbb{Z}_9$-paired right-handed neutrinos.
5. Strong CP solution: $\arg(\det M_q) = 0$ from the algebraic structure, plus a conjectured $\theta_{QCD} = 0$ from $\mathbb{Z}_9$ completeness.
6. $\mathbb{Z}_9$-invariant scalar potential with nine degenerate vacua.
7. Complete anomaly analysis with Green–Schwarz resolution.
8. Domain wall confinement via the Krauss–Wilczek gauging mechanism.

13.2 What Remains Open

Three problems define the program for the third paper in this series [2]:

1. Origin of $O(1)$ coefficients. The nine diagonal Yukawa coefficients are in $[0.49, 2.09]$ but unpredicted. If $\mathbb{Z}_9$ representation theory determines the Clebsch–Gordan coefficients of the flavon couplings, these values would be derived. This would reduce the free parameters from nine to zero.

2. Dynamical $\varepsilon = 2/9$. The VEV ratio $\varepsilon = 2/9$ is algebraically forced but not dynamically derived from the potential. A mechanism that selects this ratio automatically — perhaps through a secondary minimization condition or a connection to the $\mathbb{Z}_9$ gauge structure — would complete the model.

3. Threshold corrections and scale identification. Computing the FN threshold corrections at $\mu = \Lambda$ would test whether the conjectured FN scale $\Lambda \sim 37$ GeV (Section 12) can be confirmed dynamically, eliminating the last free dimensional parameter.

13.3 The Program So Far

[1] showed that $\mathbb{Z}_9$ arithmetic produces the right numbers: 32 predictions spanning 12 orders of magnitude at 10^{-65} significance. This paper shows that $\mathbb{Z}_9$ dynamics produces the right Lagrangian: a minimal extension of the Standard Model by one scalar field and one discrete symmetry. Together, they constitute a concrete, falsifiable framework: one axiom ($\mathbb{Z}_9$), one energy scale (m_e), one field (ϕ), zero continuous flavour symmetries, zero supersymmetry, zero extra dimensions. The Standard Model needs 19 free parameters. This framework derives them from one structure and one scale. Whether the remaining $O(1)$ coefficients can also be derived — reducing the parameter count to exactly one — is the question for [2].

Acknowledgments

Soli Deo gloria.

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A Explicit Numerical Computation

For complete reproducibility, we present the full numerical computation. All calculations can be verified with a pocket calculator.

A.1 Powers of $\varepsilon = 2/9$

A.2 Mass Matrix Eigenvalues ($c_{ij} = 1$)

With all $O(1)$ coefficients set to unity, the mass matrices are diagonalized by SVD. The singular values (in GeV) are:

n	ε^n (exact fraction)	ε^n (decimal)	$\varepsilon^n \times v$ (GeV)
0	1	1.000000	174.000
1	2/9	0.222222	38.667
2	4/81	0.049383	8.593
3	8/729	0.010974	1.909
4	16/6561	0.002438	0.4243
5	32/59049	5.419×10^{-4}	0.09429
6	64/531441	1.204×10^{-4}	0.02095
7	128/4782969	2.676×10^{-5}	0.004657
8	256/43046721	5.947×10^{-6}	0.001035

Sector	σ_1 (heaviest)	σ_2	σ_3 (lightest)
Up-type	174.4	1.87	9.2×10^{-4}
Down-type	8.65	0.0907	4.1×10^{-3}
Charged lepton	2.71	0.0877	3.4×10^{-4}

These are within a factor of ~ 2 of the measured values for all nine fermions, confirming that the charge assignments produce the correct order-of-magnitude hierarchy even before $O(1)$ coefficient fitting.

A.3 Fitted Diagonal $O(1)$ Coefficients

Sector	c_{11} (1st gen)	c_{22} (2nd gen)	c_{33} (3rd gen)	Range
Up-type (u, c, t)	2.09	0.67	0.99	[0.67, 2.09]
Down-type (d, s, b)	1.00	0.99	0.49	[0.49, 1.00]
Charged lepton (e, μ, τ)	0.49	1.12	0.93	[0.49, 1.12]

Overall range: $[0.49, 2.09]$. Mean: 1.02. Geometric mean: 0.89. Maximum deviation from unity: factor of 2.1 (up quark). All coefficients are within the standard FN naturalness criterion of $|c| \in [1/3, 3]$.

B Charge Conservation Verification

Every Yukawa coupling must conserve $\mathbb{Z}_9$ charge. For the term $c_{ij}(\phi/\Lambda)^n \bar{Q}_{L_i} H f_{R_j}$, charge conservation requires:

$$q(Q_{L_i}) + q(f_{R_j}) + n \times q(\phi) + q(H) \equiv 0 \pmod 9 \tag{44}$$

Since $q(\phi) = +1$ and $q(H) = 0$, and the flavon insertions carry charge $+n$ (from n copies of ϕ):

$$q(Q_{L_i}) + q(f_{R_j}) + n \equiv 0 \pmod 9 \implies n = -(q(Q_{L_i}) + q(f_{R_j})) \bmod 9 = (9 - q(Q_{L_i}) - q(f_{R_j})) \bmod 9 \tag{45}$$

But since we define $n = (q(Q_{L_i}) + q(f_{R_j})) \bmod 9$, the charge is carried by $\phi^\dagger$ (conjugate flavon) insertions when the sum exceeds 0. The convention is equivalent: n insertions of ϕ or $(9-n)$ insertions of $\phi^\dagger$, both carrying the same $\mathbb{Z}_9$ charge. Verification for the up-type sector (all 9 entries):

(i,j)	$q(Q_{L_i})$	$q(u_{R_j})$	Sum mod 9	Power used	Check
$(1,1)$	3	5	8	8	✓
$(1,2)$	3	1	4	4	✓
$(1,3)$	3	0	3	3	✓
$(2,1)$	2	5	7	7	✓
$(2,2)$	2	1	3	3	✓
$(2,3)$	2	0	2	2	✓
$(3,1)$	0	5	5	5	✓
$(3,2)$	0	1	1	1	✓
$(3,3)$	0	0	0	0	✓

The down-type and lepton sectors pass identically. All 27 Yukawa entries conserve $\mathbb{Z}_9$ charge by construction.

C Anomaly Computation Details

For reference, the complete anomaly computation including the cubic $\mathbb{Z}_9$ anomaly:

C.1 Charge Sums

Field Charges	$\sum q_i$	$\sum q_i$ mod 9	$\sum q_i^3$	$\sum q_i^3$ mod 9
Q_L (3, 2, 0)	5	5	35	8
u_R (5, 1, 0)	6	6	126	0
d_R (4, 3, 2)	9	0	99	0
L_L (3, 2, 0)	5	5	35	8
e_R (5, 3, 3)	11	2	179	8
ν_R (6, 7, 0)	13	4	559	1

C.2 Mixed Anomalies

$$A[SU(3)^2 \times \mathbb{Z}_9] = 2(5) + 6 + 0 = 16 \equiv 7 \pmod 9 \quad (46)$$

$$A[SU(2)^2 \times \mathbb{Z}_9] = 3(5) + 5 = 20 \equiv 2 \pmod 9 \quad (47)$$

$$A[\text{grav}^2 \times \mathbb{Z}_9] = 6(5) + 3(6) + 3(0) + 2(5) + 2 + 4 = 64 \equiv 1 \pmod 9 \quad (48)$$

C.3 Cubic Anomaly

$$A[\mathbb{Z}_9^3] = 6(35)+3(126)+3(99)+2(35)+179+559 = 210+378+297+70+179+559 = 1693 = 188\times 9+1 \equiv 1 \quad (49)$$

All four anomaly coefficients are non-zero mod 9. The Green–Schwarz mechanism cancels all anomalies simultaneously through a single axion with prescribed couplings (Section 10.3).