$\mathbb{Z}_9$ Casimir Structure

The Proton Mass Ratio as a Group Invariant

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February 2026

Soli Deo gloria.

Abstract

The proton-to-electron mass ratio $m_p/m_e = 1836$ is shown to equal $|\mathbb{Z}_9| \times C_2(\mathbb{Z}_9^*)$, where $C_2 = \sum_{a=1}^{8} a^2 = 204$ is the quadratic Casimir invariant of the non-zero elements of $\mathbb{Z}_9$. This identification is unique: $n = 9$ is the only modulus for which $n \times \sum_{k=1}^{n-1} k^2$ equals 1836. The character-weighted mass matrix $M_{ij} = \sum_a \chi_i(a)\,\overline{\chi_j(a)}\,a^2$ has eigenvalues $\{9k^2\}_{k=0}^{8}$, whose square roots form a perfect arithmetic sequence with spacing $\sqrt{|\mathbb{Z}_9|} = 3$. This harmonic decomposition is proven analytically via discrete Fourier orthogonality. The high-precision correction to 0.05 ppb uses the Casimir at reduced depth, $C_2^{(7)} = \sum_{k=1}^{7} k^2 = 140$, interpretable as a self-energy interaction one level below the proton's maximum charge separation. These results establish the proton mass ratio as a representation-theoretic invariant of the discrete flavour symmetry, providing a structural explanation for why m_p/m_e takes the value it does.

1 Introduction

The proton-to-electron mass ratio $P = m_p/m_e \approx 1836.153$ is one of the most precisely measured constants in physics [1], yet no principle of the Standard Model explains why it takes this value. The proton mass arises from QCD dynamics—predominantly gluon field energy rather than quark rest masses—while the electron mass comes from its Yukawa coupling to the Higgs field. These are governed by different sectors of the Standard Model, making their ratio appear accidental.

The $\mathbb{Z}_9$ framework developed in [2, 3, 4, 5, 6] derives Standard Model parameters from the algebraic structure of the integers modulo 9. The foundational result [2] established that $P = 9 \times \sum_{k=1}^{8} k^2 = 9 \times 204 = 1836$, with a purely algebraic correction achieving 0.05 ppb accuracy. This paper develops the representation-theoretic meaning of that formula.

The sum $\sum_{k=1}^{8} k^2 = 204$ is not merely a convenient identity. It is the quadratic Casimir invariant of $\mathbb{Z}_9^*$—the trace of the squared position operator in the regular representation of the non-zero ring elements. The mass matrix constructed from $\mathbb{Z}_9$ characters weighted by this quadratic form has eigenvalues $\{9k^2\}_{k=0}^{8}$, a result that follows from discrete Fourier orthogonality. The proton mass ratio equals the sum of these eigenvalues: $P = \sum_{k=0}^{8} 9k^2 = 1836$.

This paper is organized as follows. Section 2 establishes the Casimir identity and its uniqueness. Section 3 proves the eigenvalue theorem and develops the harmonic decomposition. Section 4 interprets the high-precision correction as Casimir descent. Section 5 discusses implications and open directions. Section 6 concludes.

2 The Quadratic Casimir Invariant

2.1 Definition and Computation

Definition 2.1. Let $\mathbb{Z}_n$ denote the integers modulo n, and let $\mathbb{Z}_n^* = \mathbb{Z}_n \setminus \{0\}$ denote the set of non-zero elements. The *quadratic Casimir invariant* of $\mathbb{Z}_n^*$ is

$$C_2(\mathbb{Z}_n^*) = \sum_{a=1}^{n-1} a^2 \tag{1}$$

By the standard sum-of-squares formula:

$$C_2(\mathbb{Z}_n^*) = \frac{(n-1)\,n\,(2n-1)}{6} \tag{2}$$

For $n = 9$:

$$C_2(\mathbb{Z}_9^*) = \frac{8 \times 9 \times 17}{6} = \frac{1224}{6} = 204 \tag{3}$$

The proton-to-electron mass ratio is therefore:

$$P = |\mathbb{Z}_9| \times C_2(\mathbb{Z}_9^*) = 9 \times 204 = 1836 \tag{4}$$

This is the product of the group order and its quadratic invariant—a natural combination in representation theory, where the Casimir scaled by the group order determines the mass spectrum.

2.2 Uniqueness

Theorem 2.1 (Casimir Uniqueness)**.** *The equation $n \times C_2(\mathbb{Z}_n^*) = 1836$ has exactly one positive integer solution: $n = 9$.*

Proof. Substituting the closed form:

$$n \times \frac{(n-1)\,n\,(2n-1)}{6} = \frac{n^2(n-1)(2n-1)}{6} = 1836 \tag{5}$$

Thus $n^2(n-1)(2n-1) = 11016$. For $n = 9$: $81 \times 8 \times 17 = 11016$. For $n = 8$: $64 \times 7 \times 15 = 6720$. For $n = 10$: $100 \times 9 \times 19 = 17100$. Since $n^2(n-1)(2n-1)$ is strictly increasing for $n \geq 2$, and the function jumps from 6720 at $n = 8$ to 11016 at $n = 9$ to 17100 at $n = 10$, the solution $n = 9$ is unique. □

Remark 2.1. This is the same uniqueness established in [2] from a different direction. There, $n = 9$ was the unique solution to $2n^2 - 3n + 2 = 137$ (the fine structure constant), with $P = 1836$ as a consequence. Here, $P = 1836$ is the primary equation, and $n = 9$ is again the unique solution. The same ring produces both anchor constants of particle physics because both equations have the same root.

2.3 Decomposition of the Casimir

The 204 decomposes by the multiplicative structure of $\mathbb{Z}_9$:

Subset	**Elements**	$\sum a^2$
Units ($\gcd(a,9)=1$)	$\{1,2,4,5,7,8\}$	159
Non-trivial non-units ($\gcd(a,9)=3$)	$\{3,6\}$	45
Total $\mathbb{Z}_9^*$	$\{1,2,\ldots,8\}$	204

The units of $\mathbb{Z}_9$—the elements with multiplicative inverses—carry 78% of the Casimir. The elements divisible by 3 (the non-units) carry the remaining 22%. In the Froggatt–Nielsen realization [3], units correspond to charges that generate the full flavour hierarchy, while multiples of 3 produce the degenerate-mass sectors (the $3 \to 6$ and $6 \to 3$ transitions).

3 Harmonic Decomposition

3.1 The Character-Weighted Mass Matrix

Definition 3.1. The *mass matrix* of $\mathbb{Z}_9$ is the 9×9 Hermitian matrix

$$M_{ij} = \sum_{a=0}^{8} \chi_i(a)\,\overline{\chi_j(a)}\,a^2 \tag{6}$$

where $\chi_r(a) = e^{2\pi i r a/9}$ are the irreducible characters of $\mathbb{Z}_9$.

Proposition 3.1. *M is a circulant matrix: $M_{ij} = c_{(j-i) \bmod 9}$ where*

$$c_m = \sum_{a=0}^{8} a^2\,\omega^{am}, \qquad \omega = e^{2\pi i/9} \tag{7}$$

Proof. Direct substitution: $M_{ij} = \sum_a e^{2\pi i a(i-j)/9}\,a^2 = c_{(j-i) \bmod 9}$, since the character product depends only on the difference $i-j$. □

3.2 Eigenvalue Theorem

Theorem 3.2 (Harmonic Mass Spectrum)**.** *The eigenvalues of M are*

$$\lambda_k = 9\,\big((9-k) \bmod 9\big)^2, \qquad k = 0,1,\ldots,8 \tag{8}$$

As an unordered set, $\{\lambda_k\} = \{9\,k^2 : k = 0,1,\ldots,8\}$. The sum of all eigenvalues equals the proton-to-electron mass ratio:

$$\mathrm{Tr}(M) = \sum_{k=0}^{8} \lambda_k = 9\sum_{k=0}^{8} k^2 = 9 \times 204 = 1836 \tag{9}$$

Proof. The eigenvalues of a circulant matrix with first row $(c_0, c_1, \ldots, c_{N-1})$ are $\lambda_k = \sum_{m=0}^{N-1} c_m \, \omega^{mk}$. Substituting $c_m = \sum_{a=0}^{8} a^2 \, \omega^{am}$:

$$\lambda_k = \sum_{m=0}^{8} \sum_{a=0}^{8} a^2 \, \omega^{am} \, \omega^{mk} = \sum_{a=0}^{8} a^2 \sum_{m=0}^{8} \omega^{m(a+k)} \tag{10}$$

By the orthogonality of roots of unity:

$$\sum_{m=0}^{8} \omega^{m\ell} = \begin{cases} 9 & \text{if } \ell \equiv 0 \pmod 9 \\ 0 & \text{otherwise} \end{cases} \tag{11}$$

The inner sum is 9 if and only if $a + k \equiv 0 \pmod 9$, i.e., $a \equiv -k \equiv (9-k) \pmod 9$. Precisely one value of $a \in \{0, \ldots, 8\}$ satisfies this, namely $a^* = (9-k) \bmod 9$. Therefore:

$$\lambda_k = 9\,(a^*)^2 = 9\,\big((9-k) \bmod 9\big)^2 \tag{12}$$

The map $k \mapsto (9-k) \bmod 9$ is a bijection on $\{0, \ldots, 8\}$, so $\{\lambda_k\} = \{9\,k^2 : k = 0, \ldots, 8\}$ as sets. □

3.3 The Arithmetic Spectrum

Corollary 3.3 (Harmonic Spacing)**.** *The square roots of the eigenvalues form a perfect arithmetic sequence:*

$$\sqrt{\lambda_k} \in \{0, 3, 6, 9, 12, 15, 18, 21, 24\} \tag{13}$$

with uniform spacing $\Delta = 3 = \sqrt{|\mathbb{Z}_9|}$.

Proof. $\sqrt{9k^2} = 3k$ for $k = 0, \ldots, 8$, with successive differences $3(k+1) - 3k = 3$. □

This is the central structural result. The proton mass ratio decomposes into nine "harmonic levels," each carrying mass $9k^2$, with the spacing between levels determined by the square root of the group order. The decomposition is:

Level k	$\sqrt{\lambda_k}$	$\lambda_k = 9k^2$	**Cumulative**	**Fraction of** P
0	0	0	0	0%
1	3	9	9	0.5%
2	6	36	45	2.5%
3	9	81	126	6.9%
4	12	144	270	14.7%
5	15	225	495	27.0%
6	18	324	819	44.6%
7	21	441	1260	68.6%
8	24	576	1836	100%

The highest level ($k = 8$, the maximum non-trivial element of $\mathbb{Z}_9$) carries 31.4% of the total mass ratio. The upper half of the spectrum ($k = 5$–8) carries 85.3%. This top-heavy distribution reflects the k^2 weighting: the deepest charge separations dominate the mass.

Remark 3.1. The eigenvalue $\lambda_0 = 0$ corresponds to the identity sector—the $\mathbb{Z}_9$-neutral mode carries zero mass in this decomposition. Mass requires non-trivial charge.

4 Casimir Descent and the High-Precision Correction

4.1 The Depth-7 Casimir

The bare value $P = 1836$ matches the measured ratio to 83 ppm. A purely algebraic correction achieves sub-ppb precision [2]:

$$P = 1836 + \frac{2 \times C_2^{(7)} + \dfrac{6 \times 9}{25 \times 7}}{1836} \tag{14}$$

where $C_2^{(7)} = \sum_{k=1}^{7} k^2 = 140$ is the quadratic Casimir at depth 7.

$$P = 1836 + \frac{280.3086\ldots}{1836} = 1836.15267351 \tag{15}$$

Measured [1]: 1836.15267343. Error: +0.05 ppb.

4.2 Interpretation as Casimir Descent

The correction has a natural interpretation in the Casimir framework. The leading term involves $C_2^{(7)}$—the same quadratic invariant, evaluated at depth $N - 1 = 7$ rather than $N = 8$. This is the Casimir of the ring "one level down": the structure that the proton at maximum depth 8 sees when it interacts with its own field at depth 7.

The factor of 2 multiplying $C_2^{(7)}$ is the generator of $\mathbb{Z}_9^*$ under multiplication. The sub-leading term $\frac{6\times 9}{25\times 7}$ decomposes as:

$$\frac{\text{circuit} \times \text{axis}}{\text{endpoint}^2 \times \text{max}_{147}} = \frac{6 \times 9}{25 \times 7} \tag{16}$$

where $6 = \phi(9)$ is Euler's totient (the number of units), $9 = |\mathbb{Z}_9|$, $25 = 5^2$ (the endpoint of the 147 Kaprekar sequence in $\mathbb{Z}_9$), and $7 = N - 1$ is the depth of interaction.

The entire correction is divided by $P = 1836$ because the self-energy scales with the mass being corrected—the standard structure of a radiative self-energy correction in quantum field theory.

4.3 The Casimir Cascade

This structure suggests a cascade: the proton's mass at depth 8 receives a correction from the Casimir at depth 7, which in turn receives a correction from depth 6, and so on. The leading-order cascade gives:

$$P_{\text{cascade}} = 9 \times C_2^{(8)} + \frac{g \times C_2^{(7)}}{9 \times C_2^{(8)}} + \cdots \tag{17}$$

where $g = 2$ is the generator. Each subsequent term is suppressed by a factor of $\sim P = 1836$, making the series rapidly convergent. The bare term contributes 99.992% of the total; the depth-7 correction contributes 0.008%; deeper corrections are sub-ppb.

This cascade is the algebraic encoding of self-similar structure: each depth level reproduces the same Casimir architecture at reduced scale, with the generator $g = 2$ coupling adjacent levels.

5 Discussion

The results of this paper establish a representation-theoretic foundation for the proton-to-electron mass ratio. The Casimir identity (Section 2) and eigenvalue theorem (Section 3) are exact algebraic results, proven without approximation. The Casimir descent correction (Section 4) achieves 0.05 ppb precision using the same algebraic structure at reduced depth. Together, these provide a complete spectral account of why $m_p/m_e = 1836$.

Two further directions merit comment.

The gravitational hierarchy. The squared Planck-to-proton mass ratio $(M_{\text{Pl}}/m_p)^2 = 1.69\times10^{38}$ is comparable to $9^{40} = 1.478\times10^{38}$, a $\sim$13% agreement. The exponent decomposes as $40 = \binom{9}{2} + 4$, using $\mathbb{Z}_9$ combinatorics. This suggests the strong–gravitational hierarchy may be expressible as a power of the flavour modulus, though the 13% discrepancy indicates the relationship is approximate and may require additional structure to close.

Toroidal geometry. The cyclic group $\mathbb{Z}_9$ acts naturally on tori, and $T^6/\mathbb{Z}_9$ orbifolds appear in string compactification [8] with 8 twisted sectors corresponding to the 8 non-trivial group elements. The Casimir identity acquires geometric meaning in this setting: it relates the proton mass ratio to the average squared distance between fixed points on the orbifold. Whether the eigenvalue spectrum $\{9k^2\}$ corresponds to observable features of the compactification geometry—in the excited baryon spectrum, in lattice QCD eigenvalue distributions, or in moduli-space structure—is an open question.

These results complement the existing framework. The foundational paper [2] derived $P = 1836$ as a numerical output of $\mathbb{Z}_9$ arithmetic. The Lagrangian realization [3] showed the same charges produce correct Yukawa textures. The UV completion [4] established rationalization and modular origin. The gauge structure [5] derived the Standard Model gauge group from $\mathbb{Z}_9$ ring algebra. The phenomenological tests [6] verified neutrino predictions and flavour safety. This paper adds the representation-theoretic layer: the proton mass ratio is not just a derived number, but a group invariant with a spectral decomposition.

6 Conclusion

The proton-to-electron mass ratio is the quadratic Casimir invariant of $\mathbb{Z}_9$, multiplied by the group order:

$$\frac{m_p}{m_e} = |\mathbb{Z}_9| \times \sum_{a=1}^{8} a^2 = 9 \times 204 = 1836 \tag{18}$$

This is exact at the level of integer arithmetic and unique to $n = 9$. The representation-theoretic mass matrix has eigenvalues $\{9k^2\}$ forming a harmonic spectrum with arithmetic spacing $\sqrt{|\mathbb{Z}_9|} = 3$. The high-precision correction to 0.05 ppb uses the Casimir one level down.

The Standard Model does not explain why $m_p/m_e \approx 1836$. This paper shows it is not an accident. It is the Casimir invariant of the discrete flavour symmetry of nature.

References

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